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BENDERS DECOMPOSITION TO SOLVE LARGE-SCALE OF THE FACILITY LOCATION PROBLEM IN TRANSPORTATION

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Abstract
In supply chain management and logistics systems, the transportation costs often represent an important part. The design of transportation network offers a great potential to reduce costs, time as well as improve service quality. Hence, determining the efficient solution for large-scale of transportation problems is an important task in the field of operations research, where the problem can be formulated as the Facility Location problem (FLP). The FLP seeks to locate a number of facilities to serve a number of customers. Systematic approach to the FLP have been studied in the operations research literature, yet the best possible result in rapid computational time is still unknown. Meanwhile, Benders decomposition is an exact algorithm that allows the solution of very large linear programming problems, quickly and optimally. In this research, we consider the Capacitated Facility Location Problem (CFLP). We seek to address the facility location strategy such that the location of hubs, the allocation of supplier/client nodes to hubs, as well as the inter-hub freight transportations, in order to achieve an efficient network design system. The main goal of the model is to find the global optimal solution of large-scale problem in reasonable computation time.
Keywords: *transportation problem, facility location problem*

INTRODUCTION

In supply chain management and logistics systems, the transportation costs often represent an important part. The design of transportation network offers a great potential to reduce costs, time, the environmental impacts as well as improve service quality[1]. Hence, determining the efficient solution for location problems is an important task in the field of operations research, where the problem can be formulated as a linear programming problem.

The Facility Location Problem (FLP) seeks to locate a number of facilities to serve a number of customers; thus, is a set of potential facility locations F, opening a facility at location $i \in F$ has an associated nonnegative fixed cost $i \in F$ and has either a limited or unlimited capacity S_i of available supply[2].

There is a set of customers or demand points D that require service; customer $j \in D$ has a demand d_j that have to be satisfied by the open facilities. If a facility at location $i \in F$ is used to satisfy part of the demand of client $j \in D$, then there is a service or transportation cost incurred, which is often proportional to the distance from i to j, as c_{ij}.

Nusantara Hasana Journal
Volume 2 No. 7 (Desember 2022), Page: 283-293
E-ISSN : 2798-1428

Systematic approach to the FLP have been studied in the operations research literature since the 1960's [3] Initial attempt at solving the problem involved intuitive heuristics, such as greedy algorithm [3]. These approaches do not provide the global optimal result, or cheapest possible solution. In fact, according to [4] the FLP is a NP-Hard problem, meaning it takes exponential time to solve because there is no known an exact algorithm that can solve it in polynomial time. Recent literatures have focused on efficiently getting an optimal global solution within several provable bound. This approach has been extensively studied in theoretical computer science, since 1960's [4][5]–[7]. Many of the techniques from the field of approximation algorithm have been successfully applied to solve the LFP, yet the best possible result in rapid computational time is still unknown.

Meanwhile, Benders decomposition is an exact algorithm that allows the solution of very large linear programming problems that have a special block structure. The strategy behind Benders decomposition can be summarized as divide-and-conquer algorithm that aims to able solve large-scale optimization problem, quickly and optimally. By applying Benders decomposition to solve the LFP, we eager is to find the global optimal solution of large-scale problem in reasonable computation time.

Problem Formulations

In this research, we consider the Capacitated Facility Location Problem (CFLP) We seek to address the facility location strategy such that the location of hubs, the allocation of supplier/client nodes to hubs, as well as the inter-hub freight transportations, in order to achieve an efficient network design system. The main goal of the model is to find the global optimal solution of large-scale problem in reasonable computation time.

Objectives

In this research, the first objective of the research is to optimize the total network cost of the FLP. To reach this objective, we propose a mixed integer linear programming model (MILP) for the CHLP with aims at minimizing the total cost for the transport network. Computational experimentations are conducted with CPLEX on the basis of a set of random instances, with size from 20s until 2000s nodes.

MATERIALS AND METHODS

Facility Location Problem

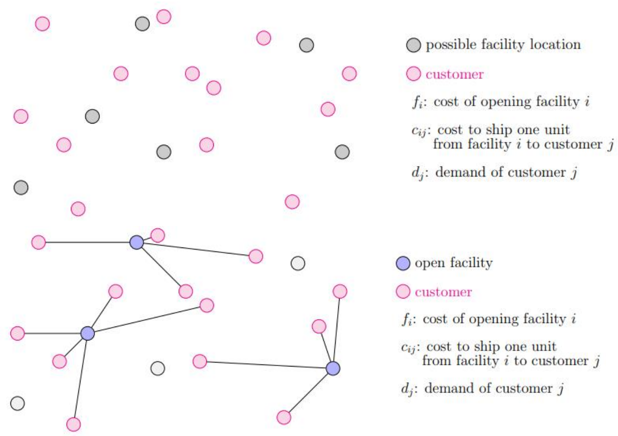
Suppose that a media company plans to place newspaper stands in a city. The company has already identified potential stand sites in a number of different neighborhoods and knows the cost of placing and maintaining a stand at each potential site[4]. Moreover, assume that the demand for the newspaper in each neighborhood of the city is exactly known. If the company wants to open any number of stands, where should they be located in order to minimize the sum of the total placing and maintaining cost and the average travelling distance of the customer[8].

The earlier question is an example of a Facility Location Problem (FLP). The FLP has been studied in the field of operations research since the 1960's [9] The FLP arise in many situations. For instance, we may consider locating warehouses, fire stations or hospitals. Basically, the FLP is characterized by four elements [9]:

Nusantara Hasana Journal
Volume 2 No. 7 (Desember 2022), Page: 283-293
E-ISSN : 2798-1428

1. **A set of locations** where facilities may be built/opened. For every location, several information, such as the cost of building or opening a facility at that location is given.
2. **A set of demand points (clients)** that have to be assigned for service to some facilities. For every client, one receives some information regarding its demand and about the costs/profits incurred if the clients be served by a certain facility.
3. **A list of requirements** to be met by the open facilities and by any assignment of demand points to facilities.
4. **A function that associates to each set of facilities** the cost/profit incurred if one would open all the facilities in the set and would assign the demand points to them such that requirements are satisfied.

The goal of the problem is then to find the set of facilities to be opened in order to optimize the given function.



(a) Problem definition (b) A possible feasible solution

Figure 1. Illustration of Facility Location Problem

Mathematical Formulation

Let m be the number of facility locations under consideration. Let n be the number of customers. We define two sets of variables:

$$y_i = \begin{cases} 1; \text{if location } i \text{ is openend} \\ 0; \text{otherwise} \end{cases}$$

$$x_{ij} = \text{ amount of material shipped from location } i \text{ to customer } j$$

Objective function of the problem is: $\min \sum_{i \in m} f_i y_i + \sum_{i \in m} \sum_{j \in n} c_{ij} x_{ij}$

Nusantara Hasana Journal
Volume 2 No. 7 (Desember 2022), Page: 283-293
E-ISSN : 2798-1428

Constraints:

a) Meet demand: $\sum_{i \in m} x_{ij} = d_j \forall\, j \in n$
b) Can only ship from open facilities: $x_{ij} \le d_j y_i, \forall\, i \in m$ and $\forall\, j \in n$

Thus, if $y_i = 0$, then we have to have $x_{ij} = 0$, $\forall\, i \in m$ and $\forall\, j \in n$ as well as if $y_i = 1$, $y_i = 1$, then the constraint becomes $x_{ij} \le d_j$.
Complete model:

$$
\begin{array}{llll}
\text{[MIP]} & \min\limits_{x,y} \sum\limits_{i \in m} f_i y_i + \sum\limits_{i \in m} \sum\limits_{j \in n} c_{ij} x_{ij} & & (1) \\
 & \text{s.t. } \sum_{i \in m} x_{ij} = d_j & \forall\, j \in n & (2) \\
 & \quad x_{ij} \le d_j y_i & \forall\, i \in m & (3) \\
 & \quad x_{ij} \ge 0 & \forall\, i \in m \text{ and } \forall\, j \in n & (4) \\
 & \quad y_i \in \{0,1\} & \forall\, i \in m & (5)
\end{array}
$$

We did not impose any capacity limits on the facilities, so this problem is known as an Incapacitated Facility Location Problem. If each location i can only ship u_i material, we get a Capacitated Facility Location Problem (CFLP), as follow:

$$
\begin{array}{llll}
\text{[MIP]} & \min\limits_{x,y} \sum\limits_{i \in m} f_i y_i + \sum\limits_{i \in m} \sum\limits_{j \in n} c_{ij} x_{ij} & & (6) \\
 & \text{s.t. } \sum_{i \in m} x_{ij} = d_j & \forall\, j \in n & (7) \\
 & \quad \sum_{i \in n} x_{ij} \le u_i & \forall\, i \in m & (8) \\
 & \quad x_{ij} \le d_j y_i & \forall\, i \in m \text{ and } \forall\, j \in n & (9) \\
 & \quad x_{ij} \ge 0 & \forall\, i \in m \text{ and } \forall\, j \in n & (10) \\
 & \quad y_i \in \{0,1\} & \forall\, i \in m & (11)
\end{array}
$$

Aggregated Model

As we known, equation (1) to (4) has $mn + n$ constraints. We can aggregate the constraints on shipping from open facilities to give a formulation with fewer constraints, such as:

$$\sum_{j \in n} x_{ij} \le M y_i \qquad \forall\, i \in m \qquad (12)$$

Where M is a large enough constant, usually call as $Big - M$. When $y_i = 0$, this constraint forces each of $x_{ij} = 0$. Note, that the M have to have large enough so that the constraint redundant when $y_i = 1$. The maximum amount we could possibly

Nusantara Hasana Journal
Volume 2 No. 7 (Desember 2022), Page: 283-293
E-ISSN : 2798-1428

ship from the location i is the sum of all the demands. Thus, it suffices to take $M = \sum_{j \in n} d_j$. Hence, the LFP model becomes:

$$\text{[MIP]} \quad \min_{x,y} \sum_{i \in m} f_i y_i + \sum_{i \in m} \sum_{j \in n} c_{ij} x_{ij} \tag{13}$$

$$\text{s.t.} \sum_{i \in m} x_{ij} = d_j \qquad \forall\, j \in n \tag{14}$$

$$\sum_{j \in n} x_{ij} \leq \left(\sum_{j \in n} d_j\right) y_i \qquad \forall\, i \in m \tag{15}$$

$$x_{ij} \geq 0 \qquad \forall\, i \in m \text{ and } \forall\, j \in n \tag{16}$$

$$y_i \in \{0,1\} \qquad \forall\, i \in m \tag{17}$$

As well as, the CLFP model becomes:

$$\text{[MIP]} \quad \min_{x,y} \sum_{i \in m} f_i y_i + \sum_{i \in m} \sum_{j \in n} c_{ij} x_{ij} \tag{18}$$

$$\text{s.t.} \sum_{i \in m} x_{ij} = d_j \qquad \forall\, j \in n \tag{19}$$

$$\sum_{i \in n} x_{ij} \leq u_i \qquad \forall\, i \in m \tag{20}$$

$$\sum_{j \in n} x_{ij} \leq \left(\sum_{j \in n} d_j\right) y_i \qquad \forall\, i \in m \text{ and } \forall\, j \in n \tag{21}$$

$$x_{ij} \geq 0 \qquad \forall\, i \in m \text{ and } \forall\, j \in n \tag{22}$$

$$y_i \in \{0,1\} \qquad \forall\, i \in m \tag{23}$$

Benders Decomposition

Consider the following general linear program, in standard form, where x is a real valued variable and z is a variable whose domain is defined by polytope $\mathbb{P}$:

$$\text{[MIP]} \quad \min_{x,y} f^T \cdot y + c^T \cdot x \tag{24}$$

$$\text{s.t. } Ax + By \geq b \tag{25}$$

$$y \in \mathcal{Y} \subseteq \mathbb{R}, x \geq 0 \tag{26}$$

If we replace the $c^T \cdot x$ with $V(y)$, we can rewrite the problem only using y variable as follows:

$$\min_{y} f^T \cdot y + V(y) \tag{27}$$

$$\text{s.t. } y \in \mathcal{Y} \subseteq \mathbb{R} \tag{28}$$

We then have the sub problem in terms of x. Do note that if the sub problem is unbounded, then the original problem is unbounded as well. Assuming it is bounded, we can calculate the value of $V(y)$ by solving the following problem:

$$[\text{PL}^{\text{P}}] \quad \min_{x} c^T \cdot x \tag{29}$$

$$\text{s.t. } Ax \geq b - B\hat{y} \tag{30}$$

$$x \geq 0 \tag{31}$$

Nusantara Hasana Journal
Volume 2 No. 7 (Desember 2022), Page: 283-293
E-ISSN : 2798-1428

If we consider u as the dual variable associated with $Ax \geq b - B\hat{y}$, we can define the dual problem of LP^P, as LP^D.

$$[LP^D] \quad \max_u (b - By)^T \cdot u \tag{32}$$
$$\text{s.t. } A^T x \geq c \tag{33}$$
$$u \geq 0 \tag{34}$$

Hence, the feasible region of LP^D is independent of y. Then, assume that for any given y, the primal problem (29) to (31) is feasible. Also, assume that the feasible region is not empty, the optimal solution for the original problem can be found by implicitly enumerating all the extreme points and rays. After that, let $\hat{u}_j, j \in J$ be the set of all extreme point, and $\hat{u}_{r}, r \in R$ be set of extreme rays, and then the original problem becomes:

$$[MIP] \quad \min_y z \tag{35}$$
$$\text{s.t. } z \geq f^T \cdot y + (b - By)^T \cdot \hat{u}_j \qquad j \in J \tag{36}$$
$$(b - B\hat{y})^T \cdot \hat{u}_r \leq 0 \qquad r \in R \tag{37}$$
$$y \in \mathcal{Y} \subseteq \mathbb{R} \tag{38}$$

Where constraint (36) is called optimality cuts because they ensure optimality of the dual problem (32) to (34), constraint (37) is called feasibility cuts because they ensure that (32) to (34) is not unbounded, thus the primal problem (29) to (31) is feasible which is what we assumed before. Thus, start with $\hat{K} \subseteq K$ and $\hat{R} \subseteq R$, then we will able to form the restricted master problem (RMP) as follow:

$$[MIP] \quad \min_y z \tag{39}$$
$$\text{s.t. } z \geq f^T \cdot y + (b - By)^T \cdot \hat{u}_j \qquad j \in \hat{J} \tag{40}$$
$$(b - B\hat{y})^T \cdot \hat{u}_r \leq 0 \qquad r \in \hat{R} \tag{41}$$
$$y \in \mathcal{Y} \subseteq \mathbb{R} \tag{42}$$

To solve RMP, we have to get a new y variable $\hat{y}$ ($y = \hat{y}$) in each iteration by solve the sub problem.

$$[LP^D] \quad \min_u (b - B\hat{y})^T \cdot u \tag{43}$$
$$\text{s.t. } A^T x \leq c \tag{44}$$
$$u \geq 0 \tag{45}$$

Nusantara Hasana Journal
Volume 2 No. 7 (Desember 2022), Page: 283-293
E-ISSN : 2798-1428

RESULTS

Applying Benders Decomposition to Facility Location Problem

As we discuss in section above that we attempt to solve the large-scale of CLFP by using Benders decomposition, in this section we shown how to transform the mathematical model of CLFP into restricted-mater problem (RMP) and sub problem, to apply in Benders decomposition. First, we can rewrite the model in its canonical form as:

$$\min_{x,y} \sum_{i \in m} \sum_{j \in n} \left(f_{ij} y_{ij} + c_{ij} x_{ij}\right) \tag{46}$$

$$\text{s.t. } \textstyle\sum_{i \in m} x_{ij} \geq d_j \qquad \forall\, j \in n \tag{47}$$

$$-\textstyle\sum_{i \in n} x_{ij} \geq -u_i \qquad \forall\, i \in m \tag{48}$$

$$-\textstyle\sum_{j \in n} x_{ij} \geq -M y_{ij} \qquad \forall\, i \in m \text{ and } \forall\, j \in n \tag{49}$$

$$x_{ij} \geq 0 \qquad \forall\, i \in m \text{ and } \forall\, j \in n \tag{50}$$

$$y_{ij} \in \{0,1\} \qquad \forall\, i \in m \text{ and } \forall\, j \in n \tag{51}$$

Thus, the Benders sub problem can write as follows:

$$\max_{\alpha_i \gamma_j \omega_{ij}} \sum_{i \in m} (-u_i)\alpha_i + \sum_{j \in n} d_j \gamma_j + \sum_{i \in m} \sum_{j \in n} \left(-M y_{ij}\right)\omega_{ij} \tag{52}$$

$$\text{s.t. } -\alpha_i + \gamma_j - \omega_{ij} \leq c_{ij} \qquad \forall\, j \in n \text{ and } \forall\, j \in n \tag{53}$$

$$\alpha_i, \gamma_j, \omega_{ij} \geq 0 \qquad \forall\, i \in m \text{ and } \forall\, j \in n \tag{54}$$

As well as, the Benders RMP of CFLP is:

$$\min_{y} z \tag{55}$$

$$\text{s.t. } z \geq \textstyle\sum_{i \in m} \sum_{j \in n} f_{ij} y_{ij} + \sum_{i \in m} (-u_i)\alpha_i + \sum_{j \in n} d_j \gamma_j + \sum_{i \in m} \sum_{j \in n} \left(-M y_{ij}\right)\omega_{ij} \tag{56}$$

$$\textstyle\sum_{i \in m} (-u_i)\alpha_i + \sum_{j \in n} d_j \gamma_j + \sum_{i \in m} \sum_{j \in n} \left(-M y_{ij}\right)\omega_{ij} \leq 0 \tag{57}$$

$$y_{ij} \in \{0,1\} \qquad \forall\, i \in m \tag{58}$$

For detail steps of solving the problem, figure 3 provide the detail algorithm of this approach as well as figure 4 provide the flowchart of the algorithm.

Nusantara Hasana Journal
Volume 2 No. 7 (Desember 2022), Page: 283-293
E-ISSN : 2798-1428

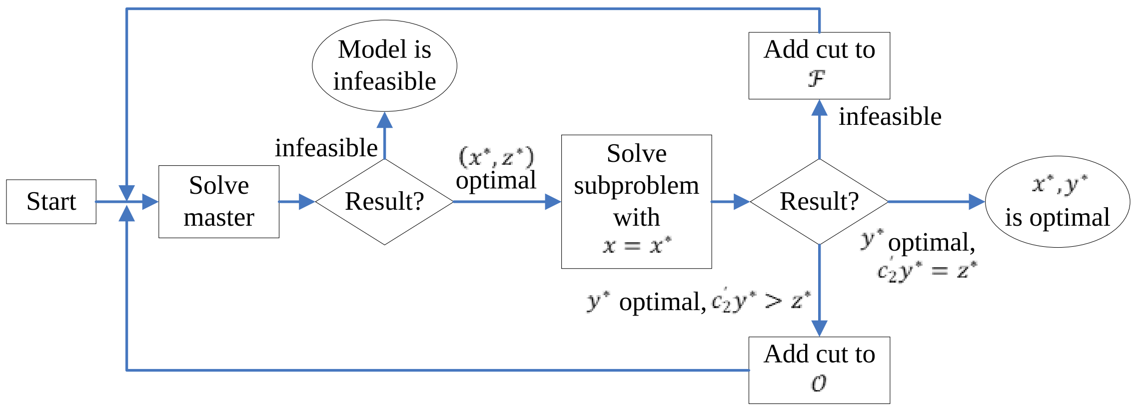


Figure 2. Flowchart of Benders Decomposition Algorithm

start procedure: Benders Decomposition Algorithm
y; $LB = -\infty$; $UB = \infty$; $k = 0$; $\leftarrow$ *initialize variables*
while $(UB - LB) > \varepsilon$ **do**
    $k = k + 1$;
    solve the subproblem
    if *unbounded* **then**
        get the extreme ray $\hat{u}_r$ and add the feasibility cut to the RMP
    if *optimal* **then**
        get the extreme point $\hat{u}_j$ and add the optimality cut to RMP
        $UB = \min\{UB, f^T \cdot \hat{y} + (b - B\hat{y})^T \cdot \hat{u}_j\}$;
end while
solve the RMP
$LB = z^k$;
end procedure

Figure 3. Benders Decomposition Algorithm

Computation Result

In this section, we will show the computation results of the bender's decomposition algorithm, by benchmarking with original linear programming method.

Table 1. Computation Result

No	Problem's Size		Computation Time (ms)	
	Supply	Demand	Linear Programming	Benders Decomposition
1	20	16	17	40
2	22	18	17	48
3	24	20	19	40
4	26	22	19	54
5	28	24	21	55
6	30	26	21	55
7	32	28	25	62

Nusantara Hasana Journal
Volume 2 No. 7 (Desember 2022), Page: 283-293
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8	38	34	31	66
9	40	36	33	82
10	44	40	37	99
11	56	47	51	99
12	66	52	107	124
13	76	57	117	165
14	86	62	115	210
15	96	6	120	131
16	100	72	150	165
17	120	82	274	269
18	200	112	1011	394
19	260	182	1507	916
20	460	226	6586	4108
21	600	446	45889	43278
22	800	580	62152	5657
23	1000	880	189611	40689
24	1200	820	194568	60174
25	2000	1800	1865662	557228
26	2200	2000	2456582	983947
27	2400	2200	6477863	4266020
28	2600	2400	8846273	5612259

Computing is done using a Personal Computer with an Intel (R) Xeon (R) CPU E5-1620 v4 @ 3.50 GHz and 16.0 GB RAM with the Windows 10 Pro 64-bit operating system. Both of the algorithm has been computed using CPLEX with C++ programming language. The comparison of both algorithms is described in the following paragraphs.

Comparison

For this research, we apply several the problem's sizes to analyze the performance of the exact algorithms. Because both algorithms are exact method, so the resulting objective value from both algorithms are same, that is the global value. Hence, to analyze the performance of the respective algorithms, we use the computation time. Following is the comparison of the computation time for the respective algorithm.

Nusantara Hasana Journal
Volume 2 No. 7 (Desember 2022), Page: 283-293
E-ISSN : 2798-1428

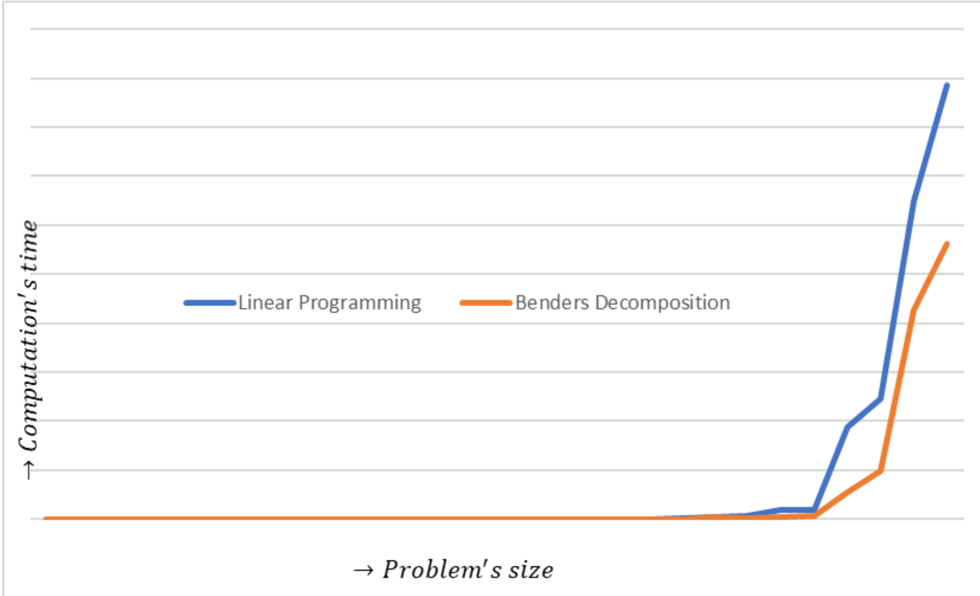

Figure 5. Comparison Computation Time of Benders Decomposition and Linear Programming

From Table 1 and Figure 5, we know that Bender decomposition is able to perform a computational process that is faster than the Linear Programming method for high problem sizes. However, for small problem sizes, the Linear Programming method, with implementation using CPLEX in C++, is able to perform a computational process that is faster than Bender decomposition. So, it can be concluded, for a large problem size, Bender decomposition is able to carry out a faster computational process than the Linear Programming method.

CONCLUSIONS

Benders decomposition is a decomposition method for solving large Mixed Integer Programming problems. Instead of solving a MIP problem that may be too large for standard solution methods all-in-one, we work with a sequence of linear and pure integer sub problems (the latter with a smaller number of constraints than the original problem).

From this research, we can conclude that Bender decomposition is able to perform a computational process that is faster than the Linear Programming method for high problem sizes. However, for small problem sizes, the Linear Programming method, with implementation using CPLEX in C++, is able to perform a computational process that is faster than Bender decomposition. So, it can be concluded, for a large problem size, Bender decomposition is able to carry out a faster computational process than the Linear Programming method.

This section may also include also include discussion on theoretical and methodological implications of findings.

REFERENCES

[1] M. Albareda-sambola, “Location Science,” *Locat. Sci.*, pp. 399–418, 2015, doi: 10.1007/978-3-319-13111-5.
[2] J. Ahn, O. De Weck, Y. Geng, and D. Klabjan, “Column generation based heuristics for a generalized location routing problem with profits arising in space exploration,” *Eur. J. Oper. Res.*, vol. 223, no. 1, pp. 47–59, 2012, doi: 10.1016/j.ejor.2012.06.018.
[3] S. Abdinnour-Helm, “A hybrid heuristic for the uncapacitated hub location problem,” *Eur. J. Oper. Res.*, vol. 106, no. 2–3, pp. 489–499, 1998, doi: 10.1016/S0377-2217(97)00286-5.
[4] R. Abyazi-Sani and R. Ghanbari, “An efficient tabu search for solving the uncapacitated single allocation hub location problem,” *Comput. Ind. Eng.*, vol. 93, pp. 99–109, 2016, doi: 10.1016/j.cie.2015.12.028.
[5] J. W. Chinneck, B. Kristjansson, and M. J. Saltzman, “Operations research and cyber-infrastructure,” *Oper. Res. Comput. Sci. Interfaces Ser.*, vol. 47, pp. 309–330, 2009, doi: 10.1007/978-0-387-88843-9.
[6] N. Asgari, M. Rajabi, M. Jamshidi, M. Khatami, and R. Z. Farahani, “A memetic algorithm for a multi-objective obnoxious waste location-routing problem: a case study,” *Ann. Oper. Res.*, vol. 250, no. 2, pp. 279–308, 2017, doi: 10.1007/s10479-016-2248-7.
[7] E. Ardjmand, G. Weckman, N. Park, P. Taherkhani, and M. Singh, “Applying genetic algorithm to a new location and routing model of hazardous materials,” *Int. J. Prod. Res.*, vol. 53, no. 3, pp. 916–928, 2015, doi: 10.1080/00207543.2014.942010.
[8] T. Aykin and G. F. Brown, “Interacting new facilities and location-allocation problems,” *Transp. Sci.*, vol. 26, no. 3, pp. 212–222, 1992, doi: 10.1287/trsc.26.3.212.
[9] M. Albareda-Sambola, J. A. Díaz, and E. Fernández, “A compact model and tight bounds for a combined location-routing problem,” *Comput. Oper. Res.*, vol. 32, no. 3, pp. 407–428, 2005, doi: 10.1016/S0305-0548(03)00245-4.